

# Predictability on Complete Financial Markets

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The following fundamental properties are proved to be true if a financial market is exhaustive: (i) Every event which is measurable by the price history at time  $T$  is independent of  $\mathcal{G}_t$  conditional on the current price history  $\mathcal{H}_t$ , where  $\mathcal{G}_t$  is a superset of  $\mathcal{H}_t$ , (ii) every event which is measurable by  $\mathcal{G}_t$  is independent of  $\mathcal{H}_T$  conditional on  $\mathcal{H}_t$ . These properties are especially useful for asset valuation, portfolio optimization and risk management. An exhaustive market with respect to  $\{\mathcal{F}_t\}$  is free of dominance and there are no free lunches with vanishing risk under  $\{\mathcal{F}_t\}$ . Moreover, it is complete with respect to every information flow which is contained in  $\{\mathcal{F}_t\}$  and the growth-optimal portfolio at time  $t$  is only determined by the past asset prices. This means any other information which is contained in  $\mathcal{F}_t$  and available to the investor at time  $t$  is irrelevant.

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## 1. Motivation

Suppose that  $\mathcal{H}_t$  is the price history on a financial market at time  $t \in [0, T]$  ( $T > 0$ ) and let  $X_T$  be some  $\mathcal{H}_T$ -measurable random vector. For example,  $X_T$  could be a vector of asset prices or of any other financial derivatives prices at time  $t$ . Now, consider a set of information  $\mathcal{G}_t \supseteq \mathcal{H}_t$ . The difference between  $\mathcal{G}_t$  and  $\mathcal{H}_t$ , i.e.,  $\mathcal{G}_t \setminus \mathcal{H}_t$  represents the information in  $\mathcal{G}_t$  which is beyond the price history  $\mathcal{H}_t$ . For example, let  $\mathcal{G}_t$  be the set of public information at time  $t$ . Then  $\mathcal{G}_t \setminus \mathcal{H}_t$  contains all public information which is available at time  $t$  but which cannot be derived from the price history at time  $t$ .

A natural requirement arising in financial econometrics and mathematical finance is

$$\mathbb{P}(X_T \leq x | \mathcal{G}_t) = \mathbb{P}(X_T \leq x | \mathcal{H}_t) \quad (1)$$

for all  $t \in [0, T]$  and every set of information  $\mathcal{G}_t$  with  $\mathcal{H}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$ . Here  $\mathcal{F}_t$  symbolizes a general set of information, e.g., the set of all private information, and  $\mathbb{P}$  denotes the physical, i.e., the “real-world” probability measure. Eq. 1 implies that the random vector  $X_T$  is  $\mathbb{P}$ -independent of  $\mathcal{G}_t$  *conditional* on the price history  $\mathcal{H}_t$ . An immediate consequence of Eq. 1 is, for example, that the conditional distribution of any asset return after time  $t$  might depend on the price history at  $t$  but not on any information  $\mathcal{G}_t$  *beyond*  $\mathcal{H}_t$ , provided  $\mathcal{G}_t$  does not exceed the general information set  $\mathcal{F}_t$ .

Another desirable property is

$$\mathbb{P}(Y_t \leq y | \mathcal{H}_T) = \mathbb{P}(Y_t \leq y | \mathcal{H}_t), \quad (2)$$

where  $Y_t$  is a  $\mathcal{G}_t$ -measurable random vector. For example, suppose that a stock company has done a balance-sheet fraud before time  $t$ . Assume that an investor is taking only the price history at time  $t$  into account and he is not aware of the fraud. It can be expected that the fraud will have an impact on the stock price up to time  $T$ , provided the latter is sufficiently far away from  $t$ , but the investor is interested to know the fraud probability *today*, i.e., at time  $t$ . It would be ideal to consider the price history in the future, i.e., at time  $T$ , since on the basis of the future price history he would get a better assessment of the fraud probability. Of course, at time  $t$ ,  $\mathcal{H}_T$  is unknown in real life. Nevertheless, Eq. 2 states that he can simply substitute  $\mathcal{H}_T$  by  $\mathcal{H}_t$ . This means all relevant information for calculating the fraud probability is already contained in the current asset prices.

In this work I will derive a sufficient condition for Eq. 1 and Eq. 2 in (simple) terms of financial mathematics and explain how this condition can be interpreted from an economic point of view. In my opinion this is essential for different reasons:

- The relationship expressed by (1) can be interpreted as a probabilistic definition of Fama's (1970) famous hypothesis that asset prices "fully reflect" the information contained in  $\mathcal{F}_t$ . For example, let  $\mathcal{F}_t$  be the set of all private information at time  $t$ . If the financial market is strong-form efficient (Fama, 1970), all private information besides the price history  $\mathcal{H}_t$  can be ignored because it is already "incorporated" in  $\mathcal{H}_t$ . Hence, if somebody aims at quantifying the conditional distribution of  $X_T$  (with respect to the *physical* measure), the weaker condition  $\mathcal{H}_t$  is as good as the stronger one  $\mathcal{F}_t$ .
- The joint distribution of future asset prices is always a function of the underlying information and in a *risky* situation (Knight, 1921), the quality of each decision cannot decrease when increasing the underlying information. Hence, every market participant tries to gather as much information as possible.<sup>1</sup> Therefore, a financial-market model must specify which kind of information is used by the economic subjects for making their investment-consumption decisions.<sup>2</sup> Eq. 1 implies that an investment decision cannot become better when using the information  $\mathcal{F}_t \setminus \mathcal{H}_t$ , provided the price history has been already taken into consideration. This means the additional information can be simply ignored by the decision makers because it would not alter the conditional distribution of the asset prices.
- Hence, in a pure investment economy where Eq. 1 is satisfied, current asset prices are unaffected by revealing  $\mathcal{F}_t$  to all market participants, provided they already know the price history  $\mathcal{H}_t$ , are rational (i.e., act according to the principles of expected utility), and there are no arbitrage opportunities under  $\mathcal{F}_t$ . For example, suppose that revealing all private information does not alter the asset prices. Then the financial market is strong-form efficient in the sense of Malkiel (1992).

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<sup>1</sup>This statement is true if the information costs are negligible (Grossman and Stiglitz, 1980). Otherwise the market participants stop searching for information when their marginal costs approach their marginal profits (Jensen, 1978).

<sup>2</sup>In most cases it is simply assumed that all market participants use the same information set and it is suggested that the conditional distribution of the asset prices is an unconditional one.

- Risk-neutral valuation of a financial asset or derivative all the same requires a specification of the underlying information. Eq. 1 guarantees that the “fair”, i.e., risk-neutral price of any contract depends on  $\mathcal{F}_t$  only through  $\mathcal{H}_t$ . This means the current value of a financial asset is only determined by past realizations of asset prices, but not by other factors like, e.g., fundamental data or insider information.
- Eq. 1 implies that it is impossible to produce a better prediction of future asset prices (or their derivatives) by using the information  $\mathcal{G}_t$  *given* that the price history  $\mathcal{H}_t$  has been already taken into consideration. This means it holds that

$$\mathbb{E}_{\mathbb{P}}(X_T | \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(X_T | \mathcal{H}_t),$$

for all  $\mathcal{G}_t$  with  $\mathcal{H}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$ , provided the conditional expectations of  $X_T$  exist and are finite. Hence, when trying to predict future asset returns, it is meaningless to use any kind of information beyond  $\mathcal{H}_t$  but within  $\mathcal{F}_t$ . By contrast, information beyond  $\mathcal{F}_t$  indeed could be useful.

- According to Fama et al. (1969) a financial market is efficient if it rapidly adjusts to new information and Eq. 2 is the probabilistic counterpart of this statement. It implies that every event  $F_t \in \mathcal{F}_t$  is *instantaneously* reflected by the asset prices. This means the evolution of asset prices after time  $t$  would not provide any additional information for calculating the probability of  $F_t$  conditional on the price history.
- The properties expressed by (1) and (2) are especially useful for risk management. Eq. 1 guarantees that for calculating the profit-loss distribution of a portfolio of risky assets it suffices to consider only the history of asset prices, whereas any other information like, e.g., individual beliefs, economic factors, political news, etc., are irrelevant provided they do not exceed the general set  $\mathcal{F}_t$  of information. Eq. 2 implies that for assessing latent risks, i.e., risks which have been already manifested but are not yet observable, it is not useful to wait for more price data, since the relevant information in  $\mathcal{F}_t$  is fully reflected by the current asset prices.

This is a non-exhaustive list of reasons for investigating the condition under which Eq. 1 and Eq. 2 become true. To the best of my knowledge this very question has not yet been discussed in the literature. The mathematical tools I use belong to martingale theory (Jacod and Shiryaev, 2002) and the key results stem from a discipline called *Enlargements of Filtrations*, which has been developed by Yor and Jeulin (1978, 1985).

For a nice overview see Jeanblanc (2010, Ch. 2) which contains a comprehensive list of references on that topic.

The contribution of this paper is a simple but substantial condition for the two equations mentioned above. It is shown that (1) and (2) are true if the financial market is *exhaustive* with respect to  $\{\mathcal{F}_t\}$ . This means that the market must be complete with respect to every information flow  $\{\mathcal{G}_t\}$  which does not exceed  $\{\mathcal{F}_t\}$ .<sup>3</sup> Moreover, I show that under these circumstances we obtain a  $\mathbb{P}$ -martingale with respect to every subfiltration of  $\{\mathcal{F}_t\}$ , after discounting the vector process  $\{S_t\}$  of original asset prices by the value of the *growth-optimal portfolio* (GOP). The GOP plays an important role in modern finance (MacLean et al., 2011; Platen and Heath, 2006). The resulting martingale property with respect to the *physical* measure confirms Samuelson's (1965) original martingale hypothesis in terms of  $\mathbb{P}$  instead of an equivalent martingale measure  $\mathbb{Q}$ .<sup>4</sup> In particular, by choosing the GOP as a numéraire, the risk-neutral valuation approach developed by Harrison and Pliska (1981) can be simply applied by taking  $\mathbb{P}$  as a martingale measure instead of  $\mathbb{Q}$ .

The quintessence of this paper is that market completeness is an essential requirement for (1) and (2) but, nevertheless, completeness only with respect to  $\{\mathcal{F}_t\}$  is not sufficient. By contrast, an exhaustive market with respect to  $\{\mathcal{F}_t\}$  is complete with respect to *every* information flow  $\{\mathcal{G}_t\}$  which is equal or less than  $\{\mathcal{F}_t\}$ . For example, let  $\{\mathcal{F}_t\}$  be the flow of private information. If the market is exhaustive with respect to  $\{\mathcal{F}_t\}$ , each payoff at time  $T$ , which is determined by public events occurring up to time  $T$ , can be replicated by an investment strategy based on the flow of public information, whereas for replicating a payoff which is only a function of the price history at time  $T$ , it is sufficient to utilize the evolution of asset prices up to time  $T$ .

## 2. Preliminary Definitions and Assumptions

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$  be a filtered probability space satisfying the usual conditions. This means  $\{\mathcal{F}_t\}$  is right-continuous and complete. Further,  $\mathcal{F}_0$  contains only  $\Omega$  as well as the  $\mathbb{P}$ -null elements of  $\mathcal{F}$ . Moreover, it is supposed that  $\mathcal{F}_T = \mathcal{F}$ . Consider a financial market with  $N + 1$  assets and let  $S_t = (S_{0t}, S_{1t}, \dots, S_{Nt})$  be the vector of asset prices at

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<sup>3</sup>The technical details will be clarified later on.

<sup>4</sup>Samuelson (1965) originally referred to futures prices and ignored interest as well as risk aversion.

time  $t \in [0, T]$ . The price process  $\{S_t\}$  is an  $\{\mathcal{F}_t\}$ -adapted  $\mathbb{R}^{N+1}$ -valued semimartingale being right-continuous with left limits (*càdlàg*).<sup>5</sup> The asset prices are assumed to be strictly positive (a.s.). In the following I will often refer to the  $\mathbb{R}^{N+1}$ -valued process  $\{P_t\}$  with  $P_t = (1, S_{1t}/S_{0t}, \dots, S_{Nt}/S_{0t})$  of *discounted* asset prices.<sup>6</sup> Finally, I apply the usual normalization  $S_{00} = 1$ .

The sequence  $\{\mathcal{F}_t\}$  can be interpreted as a flow of information evolving through time. Since  $\{S_t\}$  is adapted to  $\{\mathcal{F}_t\}$ , the latter contains at least the price history at time  $t$ . More precisely, the evolution of asset prices can be represented by the *natural filtration*  $\{\mathcal{H}_t\}$  of  $\{S_t\}$ , i.e.,  $\mathcal{H}_t$  is the  $\sigma$ -algebra generated by the evolution of asset prices between time 0 and  $t$ .<sup>7</sup> Every filtration  $\{\mathcal{G}_t\}$  with  $\mathcal{H}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$  for all  $t \in [0, T]$  will be referred to as a *subfiltration* of  $\{\mathcal{F}_t\}$ . For notational convenience I will frequently write  $\mathcal{H} = \mathcal{H}_T$  and  $\mathcal{G} = \mathcal{G}_T$ . The notation “ $X = Y$ ” for any two random variables  $X$  and  $Y$  on the measure space  $(\Omega, \mathcal{F}, \mathbb{P})$  means that  $X$  and  $Y$  are  $\mathbb{P}$ -almost surely equal. Inequalities between two random quantities shall be interpreted in the same sense. If  $X$  is a random vector and  $Y$  a scalar or a vector of the same dimension as  $X$ , then “ $X/Y$ ” is understood to mean the result of a component-wise division and has the same dimension as  $X$ .

The  $\mathbb{P}$ -expectation of a  $\mathcal{G}$ -measurable and  $\mathbb{P}$ -integrable random vector  $X$  conditional on  $\mathcal{G}_t$  is symbolized by  $E_{\mathbb{P}}(X|\mathcal{G}_t)$  and corresponds to a  $\mathcal{G}_t$ -measurable random vector  $x$  such that

$$\int_{G_t} x(\omega) d\mathbb{P}(\omega) = \int_{G_t} X(\omega) d\mathbb{P}(\omega), \quad \forall G_t \in \mathcal{G}_t.$$

The same definition applies to any other probability measure  $\mathbb{Q}$  on the measurable space  $(\Omega, \mathcal{F})$ . A  $\{\mathcal{G}_t\}$ -adapted (vector) process  $\{X_t\}$  is said to be a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$  if  $E_{\mathbb{Q}}(|X_t|) < \infty$  and  $E_{\mathbb{Q}}(X_f|\mathcal{G}_t) = X_t$  for all  $0 \leq t \leq f \leq T$ . I will say that  $\mathbb{Q}$  is a *martingale measure* on  $\mathcal{F}$  with respect to  $\{\mathcal{G}_t\}$  if and only if  $\{P_t\}$  is a  $\mathbb{Q}$ -martingale with respect to the filtration  $\{\mathcal{G}_t\}$  and  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $\mathcal{F}$ . By contrast,  $\mathbb{Q}$  is a martingale measure on  $\mathcal{G}$  with respect to  $\{\mathcal{G}_t\}$  if and only if  $\{P_t\}$  is a  $\mathbb{Q}$ -martingale with respect to the filtration  $\{\mathcal{G}_t\}$  and  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $\mathcal{G}$  but not necessarily on  $\mathcal{F}$ .

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<sup>5</sup>It is not assumed that  $\{S_t\}$  is bounded or locally bounded.

<sup>6</sup>The product of two semimartingales is also a semimartingale (Harrison and Pliska, 1981) and  $f(x) = x^{-1}$  is twice continuously differentiable, so that by Itô's Lemma  $\{P_t\}$  is a semimartingale, too.

<sup>7</sup>Since  $\{S_t\}$  is *càdlàg*, the natural filtration is right-continuous and it is also assumed that  $\{\mathcal{H}_t\}$  is complete.

Suppose that  $\mathbb{Q}$  is a probability measure being equivalent to  $\mathbb{P}$  on  $\mathcal{G}$ . From the Radon-Nikodym theorem it follows that there exists one and only one positive  $\{\mathcal{G}_t\}$ -adapted process  $\{\mathcal{L}_t^{\mathcal{G}}\}$  such that

$$\int_{G_t} d\mathbb{Q} = \int_{G_t} \mathcal{L}_t^{\mathcal{G}} d\mathbb{P}, \quad \forall G_t \in \mathcal{G}_t.$$

The random variable  $\mathcal{L}_t^{\mathcal{G}} > 0$  with  $\mathcal{L}_0^{\mathcal{G}} = 1$  corresponds to the Radon-Nikodym derivative of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  on  $\mathcal{G}_t$ . This can be interpreted as a likelihood ratio between  $\mathbb{Q}$  and  $\mathbb{P}$  and so  $\{\mathcal{L}_t^{\mathcal{G}}\}$  is referred to as a *likelihood-ratio process* with respect to  $\{\mathcal{G}_t\}$ .

It follows that  $\mathbb{E}_{\mathbb{P}}(\mathcal{L}_t^{\mathcal{G}}) = 1$  for all  $t \in [0, T]$ . Moreover, for every random vector  $X_t \in \mathcal{G}_t$  we can write

$$\int_{G_t} X_t d\mathbb{Q} = \int_{G_t} X_t \mathcal{L}_t^{\mathcal{G}} d\mathbb{P}, \quad \forall G_t \in \mathcal{G}_t,$$

i.e.,  $\mathbb{E}_{\mathbb{Q}}(X_t) = \mathbb{E}_{\mathbb{P}}(\mathcal{L}_t^{\mathcal{G}} X_t)$ . If  $\{X_t\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$  it follows that

$$\int_{G_t} X_f \mathcal{L}_f^{\mathcal{G}} d\mathbb{P} = \int_{G_t} X_f d\mathbb{Q} = \int_{G_t} X_t d\mathbb{Q} = \int_{G_t} X_t \mathcal{L}_t^{\mathcal{G}} d\mathbb{P}, \quad \forall G_t \in \mathcal{G}_t,$$

i.e.,  $\mathbb{E}_{\mathbb{P}}(\mathcal{L}_f^{\mathcal{G}} X_f | \mathcal{G}_t) = \mathcal{L}_t^{\mathcal{G}} X_t$ , so that  $\{\mathcal{L}_t^{\mathcal{G}} X_t\}$  is a  $\mathbb{P}$ -martingale with respect to  $\{\mathcal{G}_t\}$ .

A *trading strategy*  $\{H_t\}$  on  $\{\mathcal{G}_t\}$  is a  $\{\mathcal{G}_t\}$ -predictable  $\mathbb{R}^{N+1}$ -valued stochastic process which is integrable with respect to  $\{P_t\}$  (Harrison and Pliska, 1983).<sup>8</sup> The value of the strategy at time  $t$  is given by

$$V_t = H_t' P_t = V_0 + \int_0^t H_s' dP_s,^9$$

where  $V_0 = H_0' P_0$  denotes the initial value of the strategy. This means  $V_t$  evolves from *self-financing* transactions between 0 and  $t$ . The strategy  $\{H_t\}$  is called *admissible* if the value process is non-negative, i.e.,  $V_t \geq 0$  for all  $t$  with  $0 \leq t \leq T$ .<sup>10</sup>

I will use the shorthand notation  $\int H' dP = \int_0^T H_t' dP_t$  for the gain between 0 and  $T$ . An admissible strategy  $\{H_t\}$  which is such that

- (i)  $\mathbb{P}(\int H' dP \geq 0) = 1$  and
- (ii)  $\mathbb{P}(\int H' dP > 0) > 0$

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<sup>8</sup>It is clarified by Jarrow and Madan (1991) that the requirements on the process  $\{H_t\}$  which are mentioned by Harrison and Pliska (1981) are too strict. See also Remark 1.3 in Biagini (2010).

<sup>9</sup>Whenever I write  $\int_0^t H_s' dP_s$ , I mean  $\sum_{i=0}^N \int_{[0,t]} H_{is} dP_{is}$ , where  $H_{is}$  denotes the  $i$ -th component of  $H_s$ .

<sup>10</sup>From Ansel and Stricker (1994, Corollary 3.5) it follows that  $\{V_t\}$  is a local martingale if  $\{H_t\}$  is an admissible strategy.

is said to be an *arbitrage*. Moreover, an admissible strategy  $\{H_t\}$  is said to be *dominant* if

- (i)  $\mathbb{P}(\int H' dP \geq P_{iT} - P_{i0}) = 1$  and
- (ii)  $\mathbb{P}(\int H' dP > P_{iT} - P_{i0}) > 0$

for some asset  $i = 0, 1, \dots, N$ . Dominance can be interpreted as “relative arbitrage” with respect to an asset. No dominance (ND) implies no arbitrage (NA) but not vice versa. Moreover, the ND condition implies that no asset can be dominated on *any* time interval  $[s, t]$  ( $0 \leq s \leq t \leq T$ ). Otherwise, the agent could hold the corresponding asset from 0 to  $s$ , switch to the dominant strategy from  $s$  to  $t$ , switch back to the asset at time  $t$  and maintain the long position until  $T$ . This would be a dominant strategy from 0 to  $T$ , i.e., the ND condition would be violated.<sup>11</sup>

Now, consider a sequence  $\{H_{tn}\}_{n \in \mathbb{N}}$  of admissible strategies and let  $\int H'_n dP$  be the gain of the  $n$ -th strategy. It is assumed that  $\int H'_n dP$  is uniformly bounded.<sup>12</sup> Let  $G$  be a non-negative and uniformly bounded random variable with  $\mathbb{P}(G > 0) > 0$  and suppose that  $\|\int H'_n dP - G\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  (Delbaen and Schachermayer, 1997). In that case the sequence of trading strategies is said to be a *free lunch with vanishing risk* (Delbaen and Schachermayer, 1994, 1995). This is *essentially* an arbitrage (Delbaen and Schachermayer, 1997), since the maximum loss can be chosen to be arbitrarily small. If there is no free lunch with vanishing risk (NFLVR), there is NA but the converse is not true in general.

Both a free lunch with vanishing risk and a dominant strategy can be viewed as weak arbitrage opportunities. In the following I will say that there is *no weak arbitrage* (NWA) if there is NFLVR and ND. Hence, a weak arbitrage can be either a free lunch with vanishing risk, a dominant strategy or an arbitrage in the strict sense. Moreover, the *Law of One Price* (LOP) states that if  $\{V_t\}$  and  $\{W_t\}$  are two value processes such that  $V_T = W_T$ , then  $V_t = W_t$  for all  $t \in [0, T[$ . Otherwise one could create a self-financing strategy which does not cost anything at time  $t$  but yields almost surely  $|V_t - W_t| > 0$  numéraire assets at time  $T$ .<sup>13</sup> The LOP is a building block of risk-neutral valuation and thus it is tacitly accepted in this paper.

The following definition of market completeness is due to Harrison and Pliska (1981)

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<sup>11</sup>See also Jarrow and Larsson (2012) for a similar argument.

<sup>12</sup>A random variable  $X$  is uniformly bounded if and only if  $\mathbb{P}\text{-esssup}_\Omega X(\omega) < \infty$ .

<sup>13</sup>Nevertheless, since the considered strategy may be inadmissible, this is not necessarily an arbitrage.



and implies the existence of a martingale measure  $\mathbb{Q}$  on  $\mathcal{G}$ .<sup>14</sup> A *contingent claim*  $C$  is any  $\mathcal{G}$ -measurable and positive random variable such that  $X_T = C/S_{0T}$  is  $\mathbb{Q}$ -integrable. The contingent claim is *attainable* if there exists an admissible strategy  $\{H_t\}$  on  $\{\mathcal{G}_t\}$  such that the associated value process  $\{V_t\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$  and  $V_T = X_T$ . Here the martingale measure  $\mathbb{Q}$  must be equivalent to  $\mathbb{P}$  on  $\mathcal{G}$ . The financial market is said to be *complete* with respect to  $\{\mathcal{G}_t\}$  if every contingent claim is attainable. A *payoff* is any  $\mathcal{G}$ -measurable and  $\mathbb{R}$ -valued random variable  $P$  with  $X_T = P/S_{0T}$  being  $\mathbb{Q}$ -integrable. On a complete financial market, there always exists a superposition of an admissible and an inadmissible trading strategy on  $\{\mathcal{G}_t\}$  such that  $V_T = X_T$  and  $\{V_t\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$ .<sup>15</sup>

### 3. Complete and Exhaustive Markets

The *Second Fundamental Theorem of Asset Pricing* (Harrison and Pliska, 1983) will play a crucial role in the subsequent analysis:

**Theorem 1** (Second Fundamental Theorem of Asset Pricing). *Suppose that there exists a martingale measure  $\mathbb{Q}$  on  $\mathcal{G}$  with respect to  $\{\mathcal{G}_t\}$ . Then the following assertions are equivalent:*

- *The financial market is complete with respect to  $\{\mathcal{G}_t\}$ .*
- *The martingale measure  $\mathbb{Q}$  is unique on  $\mathcal{G}$ .*
- *Every  $\mathbb{Q}$ -martingale  $\{X_t\}$  with respect to  $\{\mathcal{G}_t\}$  can be represented by*

$$X_t = X_0 + \int_0^t H'_s dP_s,$$

*where  $\{H_t\}$  is a trading strategy on  $\{\mathcal{G}_t\}$ .*

The Second Fundamental Theorem of Asset Pricing requires a martingale measure  $\mathbb{Q}$ . Due to the First Fundamental Theorem of Asset Pricing for unbounded price processes (Delbaen and Schachermayer, 1998), there is NFLVR under  $\{\mathcal{G}_t\}$  if and only if  $\{P_t\}$  is a

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<sup>14</sup>Battig and Jarrow (1999) generalize the notion of market completeness by allowing complete markets without a martingale measure. I will not use that definition because the existence of a martingale measure is a natural requirement for risk-neutral valuation.

<sup>15</sup>The first strategy attains  $x + \max\{P, 0\} > 0$ , whereas the second strategy attains  $x - \min\{P, 0\} > 0$  for some arbitrary real number  $x > 0$ . The superposition is given by the first minus the second strategy.

$\Sigma$ -martingale under  $\{\mathcal{G}_t\}$ .<sup>16</sup> Every local martingale is a  $\Sigma$ -martingale. Moreover, every  $\Sigma$ -martingale which is bounded from below is a local martingale (Jacod and Shiryaev, 2002, p. 216). Since the discounted asset prices are positive,  $\{P_t\}$  is a local martingale whenever it is a  $\Sigma$ -martingale. This means in the present context it is not necessary to distinguish between the terms “ $\Sigma$ -martingale” and “local martingale”. The problem is that the First Fundamental Theorem of Asset Pricing only guarantees the existence of a *local* but not a strict martingale measure. Nonetheless, by the Third Fundamental Theorem of Asset Pricing (Jarrow, 2012) it follows that  $\{P_t\}$  is a strict martingale with respect to  $\{\mathcal{G}_t\}$  if and only if there is NWA under the flow of information  $\{\mathcal{G}_t\}$ . Further, due to Jarrow and Larsson (2012) there exists a pure exchange economy where  $\{P_t\}$  is an equilibrium-price process under the information flow  $\{\mathcal{G}_t\}$  if and only if there is NWA under  $\{\mathcal{G}_t\}$ .<sup>17</sup> Hence, the absence of weak arbitrage, i.e., the existence of a martingale measure  $\mathbb{Q}$  is an essential requirement not only for risk-neutral valuation but also for a market equilibrium.

**Proposition 1.** *Suppose that there exists a martingale measure  $\mathbb{Q}$  on  $\mathcal{F}$  with respect to  $\{\mathcal{G}_t\}$ . Then  $\mathbb{Q}$  is a martingale measure on  $\mathcal{F}$  with respect to  $\{\mathcal{H}_t\}$ .*

Proof:  $E_{\mathbb{Q}}(P_f | \mathcal{H}_t) = E_{\mathbb{Q}}[E_{\mathbb{Q}}(P_f | \mathcal{G}_t) | \mathcal{H}_t] = E_{\mathbb{Q}}(P_t | \mathcal{H}_t) = P_t$ . Q.E.D.

**Definition 1** (Exhaustive market). *A financial market is said to be exhaustive with respect to  $\{\mathcal{F}_t\}$  if and only if there exists one and only one martingale measure  $\mathbb{Q}$  on  $\mathcal{F}$  with respect to every subfiltration of  $\{\mathcal{F}_t\}$ .*

The following propositions contain two fundamental properties of exhaustive markets.

**Proposition 2.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$ . Then there is NWA under every subfiltration of  $\{\mathcal{F}_t\}$ .*

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<sup>16</sup>The vector process  $\{Y_t\}$  is said to be a  $\Sigma$ -martingale with respect to  $\{\mathcal{G}_t\}$  if  $Y_t$  can be written as  $Y_t = Y_0 + \int_0^t H_s dX_s$ , where  $\{H_t\}$  is an  $\{X_t\}$ -integrable  $\{\mathcal{G}_t\}$ -predictable process and  $\{X_t\}$  is a local  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$ . For other characterizations of  $\Sigma$ -martingales see Émery (1980, Proposition 2) as well as Jacod and Shiryaev (2002, Definition 6.33 and Theorem 6.41).

<sup>17</sup>More precisely,  $\{P_t\}$  is an equilibrium-price process if (i) the investment-consumption decisions of the economic subjects are optimal and (ii) all (i.e., the security and the commodity) markets clear with  $\{P_t\}$ .

Proof: Due to the Third Fundamental Theorem of Asset Pricing (Jarrow, 2012) there is NWA under any subfiltration  $\{\mathcal{G}_t\}$  if and only if  $\{P_t\}$  is a martingale with respect to  $\{\mathcal{G}_t\}$ , where the martingale measure is equivalent to  $\mathcal{G}$ . Since the market is exhaustive with respect to  $\{\mathcal{F}_t\}$ , there exists a martingale measure  $\mathbb{Q}$  which is equivalent on  $\mathcal{F}$  and thus on  $\mathcal{G} \subseteq \mathcal{F}$ . Q.E.D.

Hence, a market which is exhaustive with respect to  $\{\mathcal{F}_t\}$  is efficient with respect  $\{\mathcal{F}_t\}$  in the sense of Jarrow and Larsson (2012).

**Proposition 3.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$ . Then it is complete with respect to every subfiltration of  $\{\mathcal{F}_t\}$ .*

Proof: Let  $\{\mathcal{G}_t\}$  be any subfiltration of  $\{\mathcal{F}_t\}$ . Since the market is exhaustive with respect to  $\{\mathcal{F}_t\}$ , there exists one and only one martingale measure  $\mathbb{Q}$  on  $\mathcal{F}$  with respect to  $\{\mathcal{G}_t\}$ . Suppose that there exists a martingale measure  $\mathbb{Q}'$  on  $\mathcal{G}$ . Then  $\mathbb{Q}'$  can be extended to any probability measure on  $\mathcal{F}$  being equivalent to  $\mathbb{P}$  on  $\mathcal{F}$ . Because the market is exhaustive, it must hold that  $\mathbb{Q}' = \mathbb{Q}$ . Due to the Second Fundamental Theorem of Asset Pricing the market is complete with respect to  $\{\mathcal{G}_t\}$ . Q.E.D.

This means if the market is exhaustive, every  $\mathcal{G}$ -measurable payoff  $P$  can be replicated on the basis of the information flow  $\{\mathcal{G}_t\}$ , i.e., the broader information flow  $\{\mathcal{F}_t\}$  is not necessary. By contrast, market completeness with respect to  $\{\mathcal{F}_t\}$  only guarantees that  $P$  can be replicated by a trading strategy on  $\{\mathcal{F}_t\}$ . Thus  $\{\mathcal{F}_t\}$  might be a necessary information flow although the payoff by itself is only determined by  $\{\mathcal{G}_t\}$ .

**Theorem 2.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$ . Let  $\{\mathcal{L}_t^{\mathcal{H}}\}$  and  $\{\mathcal{L}_t^{\mathcal{F}}\}$  be the likelihood-ratio processes with respect to  $\{\mathcal{H}_t\}$  and  $\{\mathcal{F}_t\}$ . Then  $\mathcal{L}_t^{\mathcal{H}} = \mathcal{L}_t^{\mathcal{F}} > 0$  for all  $t \in [0, T]$ .*

Proof: Consider the  $\{\mathcal{F}_t\}$ -adapted process  $\{\mathcal{M}_t\}$  with

$$\mathcal{M}_t = \frac{\mathcal{L}_t^{\mathcal{H}}}{\mathcal{L}_t^{\mathcal{F}}}, \quad \forall t \in [0, T].$$

Since  $\mathbb{P}$  and  $\mathbb{Q}$  are equivalent on  $\mathcal{F}$ , it follows that  $\mathcal{L}_t^{\mathcal{F}}, \mathcal{L}_t^{\mathcal{H}} > 0$  and thus  $\mathcal{M}_t > 0$ . Now, we can define a probability measure  $\mathbb{Q}'$  by

$$\mathbb{Q}'(F_t) = \int_{F_t} \mathcal{M}_t d\mathbb{Q}, \quad \forall F_t \in \mathcal{F}_t, \forall t \in [0, T].$$

Since  $\mathcal{M}_t > 0$  it follows that  $\mathbb{Q}'$  is equivalent to  $\mathbb{Q}$  on  $\mathcal{F}$ . Moreover, we have that

$$\mathbb{Q}'(H_t) = \int_{H_t} \mathcal{M}_t d\mathbb{Q} = \int_{H_t} \frac{\mathcal{L}_t^{\mathcal{H}}}{\mathcal{L}_t^{\mathcal{F}}} \mathcal{L}_t^{\mathcal{F}} d\mathbb{P} = \int_{H_t} \mathcal{L}_t^{\mathcal{H}} d\mathbb{P} = \int_{H_t} d\mathbb{Q} = \mathbb{Q}(H_t)$$

for all  $H_t \in \mathcal{H}_t$  and  $t \in [0, T]$ . This means  $\mathbb{Q}' = \mathbb{Q}$  on  $\mathcal{H}$  and thus  $\{P_t\}$  is a  $\mathbb{Q}'$ -martingale with respect to  $\{\mathcal{H}_t\}$ . Because the market is exhaustive and  $\{\mathcal{H}_t\}$  is a subfiltration of  $\{\mathcal{F}_t\}$ , it follows that  $\mathbb{Q}' = \mathbb{Q}$  on  $\mathcal{F}$ . Hence,

$$1 = \mathbb{E}_{\mathbb{Q}}(1 | \mathcal{F}_t) = \mathcal{M}_t = \frac{\mathcal{L}_t^{\mathcal{H}}}{\mathcal{L}_t^{\mathcal{F}}},$$

i.e.,  $\mathcal{L}_t^{\mathcal{H}} = \mathcal{L}_t^{\mathcal{F}}$  for all  $t \in [0, T]$ .

Q.E.D.

The filtration  $\{\mathcal{H}_t\}$  is said to be *immersed* in  $\{\mathcal{F}_t\}$  with respect to  $\mathbb{Q}$  if every square-integrable  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{H}_t\}$  is a square-integrable  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$  (Jeanblanc, 2010, p. 16).<sup>18</sup> The statement that “ $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$ ” is often referred to as the  $\mathcal{H}$ -hypothesis (Brémaud and Yor, 1978). A number of sufficient and necessary conditions for the  $\mathcal{H}$ -hypothesis can be found in the literature (Jeanblanc, 2010, Section 2.1). The  $\mathcal{H}$ -hypothesis is frequently used in the area of credit risk (Bielecki and Rutkowski, 2002; Elliott et al., 2000). For example,  $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$  if and only if every local  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{H}_t\}$  is a local  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$  (Jeanblanc, 2010, Proposition 2.1.1).

**Theorem 3.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$ . Then  $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$  both with respect to the unique martingale measure  $\mathbb{Q}$  on  $\mathcal{F}$  and with respect to the physical measure  $\mathbb{P}$  on  $\mathcal{F}$ .*

*Proof:* Consider a square-integrable  $\mathbb{Q}$ -martingale  $\{X_t\}$  with respect to  $\{\mathcal{H}_t\}$ . From Proposition 3 it follows that the market is complete with respect to  $\{\mathcal{H}_t\}$ . This means  $\{X_t\}$  can be represented by a trading strategy on  $\{\mathcal{H}_t\}$  and let  $\{V_t^{\mathcal{H}}\}$  be the associated value process such that  $V_t^{\mathcal{H}} = X_t$  for all  $t \in [0, T]$ . Moreover, due to Proposition 3 the market is also complete with respect to  $\{\mathcal{F}_t\}$  and so the outcome  $X_T \in \mathcal{H}_T \subseteq \mathcal{F}_T$  can be replicated by a trading strategy on  $\{\mathcal{F}_t\}$ , whose value process  $\{V_t^{\mathcal{F}}\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$ . This means  $\mathbb{E}_{\mathbb{Q}}(V_T^{\mathcal{F}} | \mathcal{F}_t) = V_t^{\mathcal{F}}$  and due to the LOP we have that

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<sup>18</sup>Every square-integrable martingale with respect to  $\{\mathcal{H}_t\}$  is  $\{\mathcal{H}_t\}$ -adapted by definition. Hence, the same martingale is still square integrable after an enlargement of the filtration.

$V_t^{\mathcal{F}} = V_t^{\mathcal{H}} = X_t$  for all  $t \in [0, T]$ . Hence,

$$\mathbb{E}_{\mathbb{Q}}(X_f | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(V_f^{\mathcal{F}} | \mathcal{F}_t) = V_t^{\mathcal{F}} = X_t,$$

i.e.,  $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$  with respect to  $\mathbb{Q}$ . Moreover, from Theorem 2 together with Proposition 2.1.4 in Jeanblanc (2010) it follows that  $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$  also with respect to the physical measure  $\mathbb{P}$ . Q.E.D.

The following theorem is the main result of this paper.

**Theorem 4.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$ . Let  $X_T$  be an  $\mathcal{H}_T$ -measurable and  $Y_t$  a  $\mathcal{G}_t$ -measurable random vector. Then*

$$(i) \quad \mathbb{P}(X_T \leq x | \mathcal{G}_t) = \mathbb{P}(X_T \leq x | \mathcal{H}_t) \text{ and}$$

$$(ii) \quad \mathbb{P}(Y_t \leq y | \mathcal{H}_T) = \mathbb{P}(Y_t \leq y | \mathcal{H}_t)$$

for all  $t \in [0, T]$  and every subfiltration  $\{\mathcal{G}_t\}$  of  $\{\mathcal{F}_t\}$ .

Proof: Due to Theorem 3,  $\{\mathcal{H}_t\}$  is immersed in  $\{\mathcal{F}_t\}$  with respect to the physical measure  $\mathbb{P}$ . According to Proposition 2.1.1 in Jeanblanc (2010),  $X_T$  is conditionally  $\mathbb{P}$ -independent of  $\mathcal{F}_t$  given  $\mathcal{H}_t$ . Hence, the first statement is an immediate consequence of the law of total probability. Moreover, from Proposition 2.1.1 in Jeanblanc (2010) it follows also that  $\mathbb{E}_{\mathbb{P}}(Z | \mathcal{H}_T) = \mathbb{E}_{\mathbb{P}}(Z | \mathcal{H}_t)$  for every  $\mathbb{P}$ -integrable  $Z \in \mathcal{G}_t$ . By setting  $Z = \mathbb{1}_{\{Y_t \leq y\}}$ , where  $\mathbb{1}_{\{\cdot\}}$  denotes the indicator function, we obtain the second statement. Q.E.D.

**Proposition 4.** *Suppose that the financial market is exhaustive with respect to  $\{\mathcal{F}_t\}$  and let  $\mathbb{Q}$  be the unique martingale measure on  $\mathcal{F}$ . Further, let  $\{\mathcal{L}_t^{\mathcal{F}}\}$  be the likelihood-ratio process with respect to  $\{\mathcal{F}_t\}$  and  $\{\mathcal{G}_t\}$  any subfiltration of  $\{\mathcal{F}_t\}$ . Then  $\{\mathcal{L}_t^{\mathcal{F}}\}$  is a  $\mathbb{P}$ -martingale with respect to  $\{\mathcal{G}_t\}$  and  $\{(\mathcal{L}_t^{\mathcal{F}})^{-1}\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$ .*

Proof: From  $\mathcal{L}_t^{\mathcal{F}} > 0$  and

$$\int_{\Omega} \mathcal{L}_t^{\mathcal{F}} d\mathbb{P} = \int_{\Omega} d\mathbb{Q} = 1 < \infty$$

it follows that  $\{\mathcal{L}_t^{\mathcal{F}}\}$  is  $\mathbb{P}$ -integrable. Moreover,

$$\int_{G_t} \mathcal{L}_t^{\mathcal{F}} d\mathbb{P} = \int_{G_t} \mathcal{L}_f^{\mathcal{F}} d\mathbb{P}, \quad \forall G_t \in \mathcal{G}_t \subseteq \mathcal{F}_t$$

and from Theorem 2 we know that  $\mathcal{L}_t^{\mathcal{F}} = \mathcal{L}_t^{\mathcal{H}} \in \mathcal{G}_t$ . This means  $\mathcal{L}_t^{\mathcal{F}} = \mathbb{E}_{\mathbb{P}}(\mathcal{L}_t^{\mathcal{F}} | \mathcal{G}_t)$ , i.e.,  $\{\mathcal{L}_t^{\mathcal{F}}\}$  is a  $\mathbb{P}$ -martingale with respect to  $\{\mathcal{G}_t\}$ . Further, since  $(\mathcal{L}_t^{\mathcal{F}})^{-1} > 0$  and

$$\int_{\Omega} (\mathcal{L}_t^{\mathcal{F}})^{-1} d\mathbb{Q} = \int_{\Omega} (\mathcal{L}_t^{\mathcal{F}})^{-1} \mathcal{L}_t^{\mathcal{F}} d\mathbb{P} = \int_{\Omega} d\mathbb{P} = 1 < \infty,$$

the inverse likelihood-ratio process  $\{(\mathcal{L}_t^{\mathcal{F}})^{-1}\}$  is  $\mathbb{Q}$ -integrable. Proposition B.41 in Björk (2009) leads to

$$\mathbb{E}_{\mathbb{Q}} \left[ (\mathcal{L}_t^{\mathcal{F}})^{-1} | \mathcal{G}_t \right] = \frac{1}{\mathbb{E}_{\mathbb{P}}(\mathcal{L}_t^{\mathcal{F}} | \mathcal{G}_t)} = (\mathcal{L}_t^{\mathcal{F}})^{-1}.$$

Hence,  $\{(\mathcal{L}_t^{\mathcal{F}})^{-1}\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{G}_t\}$ . Q.E.D.

Let  $\{V_t\}$  be the value process of an admissible strategy on  $\{\mathcal{F}_t\}$  with  $V_t > 0$  for all  $t \in [0, T]$ .<sup>19</sup> The value process  $\{V_t\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$ .<sup>20</sup> Further, the random variable  $\log(V_T/V_t)$  represents the log-return on the strategy from time  $t$  to  $T$  and

$$\log \frac{V_T}{V_t} - \log \frac{(\mathcal{L}_T^{\mathcal{F}})^{-1}}{(\mathcal{L}_t^{\mathcal{F}})^{-1}} = \log \frac{\mathcal{L}_T^{\mathcal{F}} V_T}{\mathcal{L}_t^{\mathcal{F}} V_t}$$

is the difference between the value process and the inverse likelihood-ratio process in terms of the log-return. From Jensen's inequality it follows that

$$\mathbb{E}_{\mathbb{P}} \left( \log \frac{\mathcal{L}_T^{\mathcal{F}} V_T}{\mathcal{L}_t^{\mathcal{F}} V_t} | \mathcal{F}_t \right) \leq \log \frac{\mathbb{E}_{\mathbb{P}}(\mathcal{L}_T^{\mathcal{F}} V_T | \mathcal{F}_t)}{\mathcal{L}_t^{\mathcal{F}} V_t} = 0. \quad (3)$$

From Proposition 4 we know that the inverse likelihood-ratio process  $\{(\mathcal{L}_t^{\mathcal{F}})^{-1}\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$  and due to the Second Fundamental Theorem of Asset Pricing, there exists a trading strategy on  $\{\mathcal{F}_t\}$  which is able to replicate the inverse likelihood-ratio process. Eq. 3 confirms that this is the *growth-optimal strategy* on  $\{\mathcal{F}_t\}$ . If the market is exhaustive with respect to  $\{\mathcal{F}_t\}$ , Theorem 2 implies that  $(\mathcal{L}_t^{\mathcal{F}})^{-1} = (\mathcal{L}_t^{\mathcal{H}})^{-1}$ , i.e., the growth-optimal strategy on  $\{\mathcal{F}_t\}$  in fact is  $\{\mathcal{H}_t\}$ -predictable. This means the broader flow of information  $\{\mathcal{F}_t\}$  is not useful for the growth-optimal strategy if this is already based on the evolution of asset prices.

It is worth emphasizing that the value process of the growth-optimal portfolio is *unique* with respect every subfiltration of  $\{\mathcal{F}_t\}$  if the market is exhaustive with respect to  $\{\mathcal{F}_t\}$ .

<sup>19</sup>This guarantees that  $\log V_t$  is well-defined for all  $t \in [0, T]$ .

<sup>20</sup>The final value of  $\{V_t\}$ , i.e.,  $V_T$  can be attained by a trading strategy on  $\{\mathcal{F}_t\}$  and let  $\{W_t\}$  with  $W_T = V_T$  be the associated value process. Moreover,  $\{W_t\}$  is a  $\mathbb{Q}$ -martingale with respect to  $\{\mathcal{F}_t\}$  and due to the LOP we have that  $V_t = W_t$  for all  $t \in [0, T]$ .

Moreover, it is easy to see that  $S_{0t}(\mathcal{L}_t^{\mathcal{H}})^{-1}$  is the value of the growth-optimal portfolio with respect to the *original* asset prices, which are given by  $S_t$ .<sup>21</sup> In the following I will write  $\Pi_t = S_{0t}(\mathcal{L}_t^{\mathcal{H}})^{-1}$  for notational convenience.<sup>22</sup> Since

$$\mathbb{E}_{\mathbb{P}}(\Pi_f^{-1} S_f | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathcal{L}_f^{\mathcal{F}} P_f | \mathcal{F}_t) = \mathcal{L}_t^{\mathcal{F}} P_t = \Pi_t^{-1} S_t,$$

we can simply define the vector  $Q_t = \Pi_t^{-1} S_t$  of asset prices which have been discounted by the growth-optimal portfolio. The discounted price process  $\{Q_t\}$  is  $\{\mathcal{H}_t\}$ -adapted and so it follows that

$$\mathbb{E}_{\mathbb{P}}(Q_f | \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}(Q_f | \mathcal{F}_t) | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}(Q_t | \mathcal{G}_t) = Q_t.$$

This means  $\{Q_t\}$  is a  $\mathbb{P}$ -martingale with respect to every subfiltration of  $\{\mathcal{F}_t\}$ . Hence, after discounting the original asset prices by the growth-optimal portfolio value instead of  $S_{0t}$ , the risk-neutral valuation approach developed by Harrison and Pliska (1981) can be simply applied by taking  $\mathbb{P}$  as a martingale measure instead of  $\mathbb{Q}$  (Platen, 2006; Platen and Heath, 2006, Ch. 9).

For these reasons the growth-optimal portfolio is a canonical *numéraire portfolio* (Becherer, 2001; Long, 1990). Moreover, it follows that

$$\mathbb{E}_{\mathbb{P}}(R_{t,f} | \mathcal{G}_t) = \mathbf{0}$$

with  $R_{t,f} = Q_f / Q_t - \mathbf{1}$  for all  $0 \leq t \leq f \leq T$ .<sup>23</sup> This means on an exhaustive market with respect to  $\{\mathcal{F}_t\}$  it is not possible to predict asset returns on the basis of any subfiltration of  $\{\mathcal{F}_t\}$  *after discounting the original asset prices by the growth-optimal portfolio value*.

## 4. Conclusion

I derive a simple condition for two fundamental properties of a financial market:

- (i) Every event which is measurable by the price history at time  $T$  is independent of  $\mathcal{G}_t$  conditional on the current price history  $\mathcal{H}_t$ , where  $\mathcal{G}_t$  is a superset of  $\mathcal{H}_t$  and

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<sup>21</sup>The growth-optimal portfolio maximizes the drift rate of  $\{\log V_t\}$  at time  $t$  (Karatzas and Kardaras, 2007).

Here  $V_t = H_t' P_t = H_t' S_t / S_{0t}$  and because of  $d \log V_t = d \log H_t' S_t - d \log S_{0t}$ , the numéraire asset is void for the growth-optimal strategy.

<sup>22</sup>Due to the chosen normalization  $S_{00} = 1$  and the fact that  $\mathcal{L}_0^{\mathcal{H}} = 1$  it holds that  $\Pi_0 = 1$ .

<sup>23</sup>Here  $\mathbf{0}$  denotes a vector of zeros and  $\mathbf{1}$  is a vector of ones.

(ii) every event which is measurable by  $\mathcal{G}_t$  is independent of  $\mathcal{H}_T$  conditional on  $\mathcal{H}_t$ .

These properties have several implications which are especially useful in the context of asset valuation, portfolio optimization and risk management. It turns out that market completeness is an essential requirement for the desired properties (i) and (ii) but it is not sufficient to assume completeness only with respect to  $\{\mathcal{F}_t\}$ . The sufficient condition is *market exhaustivity*. If a financial market is exhaustive with respect to a general flow of information  $\{\mathcal{F}_t\}$ , it is complete with respect to every flow of information which is contained in  $\{\mathcal{F}_t\}$ . An exhaustive market is always free of weak-arbitrage opportunities, i.e., there is ND and NFLVR. Moreover, the GOP turns out to be identical under every subfiltration of  $\{\mathcal{F}_t\}$ . The GOP can serve as a numéraire portfolio, i.e., by discounting all asset prices by the GOP value it is possible to apply the risk-neutral valuation approach on the basis of the physical measure  $\mathbb{P}$  instead of some martingale measure  $\mathbb{Q}$ .

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